Close Tue: 15.4 (finish early)
Close Thu: 15.5
Note: I won't ask about 15.5 (center of mass) on exam 2, but use this homework for more practice with double integrals.
Office Hours Today: 1:30-3:00pm (Com B-006)

Exam 2, May 16th

13.3/4: **Derivatives of 3D curves**
Curvature, Arc Length, TNB-Frame, Normal Plane, Osculating Plane, Velocity, Acceleration

14.1/3/4/7: **Partial derivatives**
Level Curves, Domain, Partials, Tangent Plane, local max/min (2nd deriv. test), global max/min, applied max/min

15.1-15.4: **Double integrals**
general regions (top/bot, left/right), reversing order, polar

***Entry Task*:** How do you start these HW questions?

HW 15.4/9: Find the volume enclosed by
$-x^2 - y^2 + z^2 = 22$ and $z = 5$.

HW 15.4/10: Find the volume above the upper cone $z = \sqrt{x^2 + y^2}$ and below
$x^2 + y^2 + z^2 = 81$

15.4 Double Integrals over Polar Regions

Recall:

θ = angle measured from positive x-axis

r = distance from origin

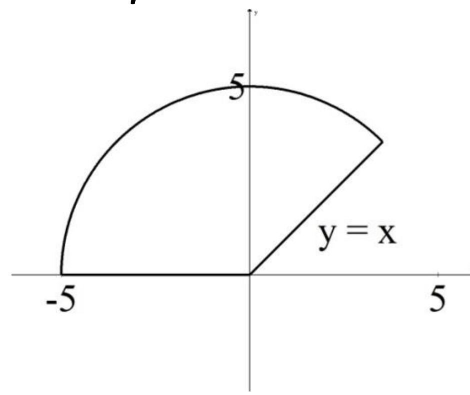
$x = r\cos(\theta)$, $y = r\sin(\theta)$, $x^2 + y^2 = r^2$

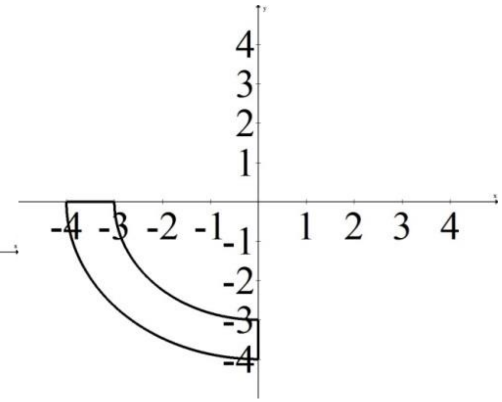
To set up a double integral in polar we will:

1. Describing the region in polar
2. Replace "x" by "r cos(θ)"
3. Replace "y" by "r sin(θ)"
4. Replace "dA" by "r dr dθ"

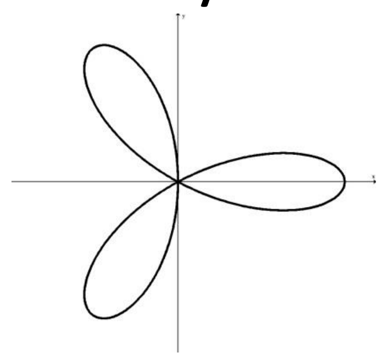
Step 1: Describing regions in polar.

Examples: Describe the regions

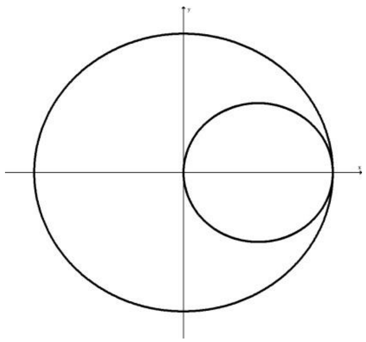




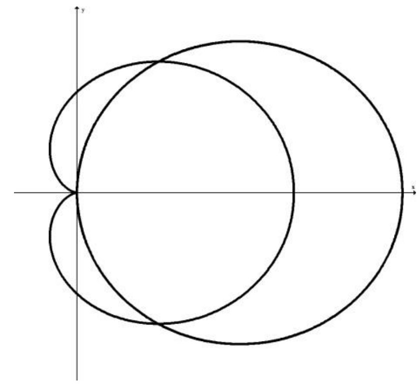
HW 15.4/5: One loop of $r = 6\cos(3\theta)$.



HW 15.4/4: Region in the first quadrant between the circles $x^2 + y^2 = 16$ and $x^2 + y^2 = 4x$.



HW 15.4/7:
Describe the region inside $r = 1+\cos(\theta)$ and outside $r = 3\cos(\theta)$.



Step 2: Set up your integral in polar.

Conceptual notes:

Cartesian

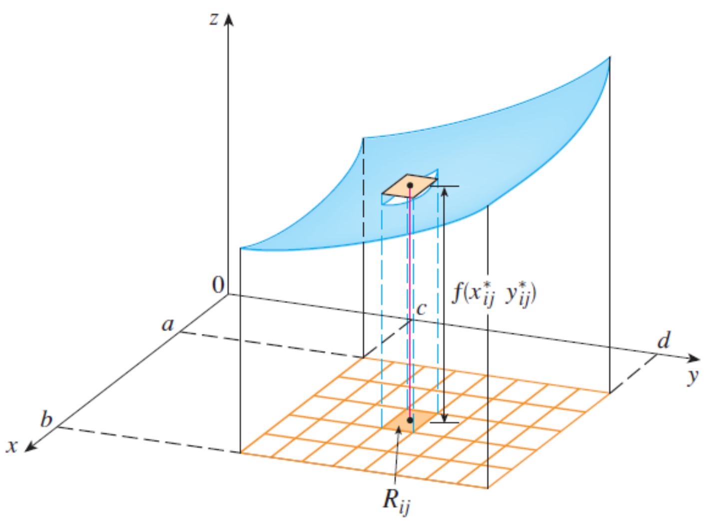
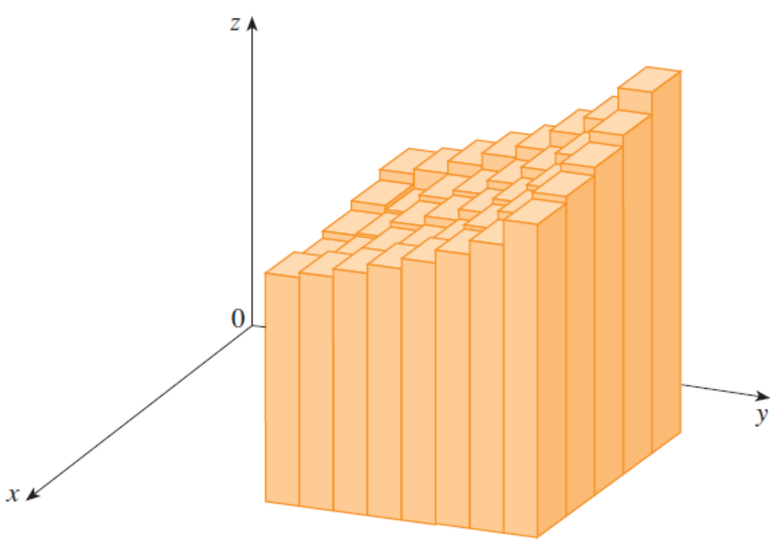
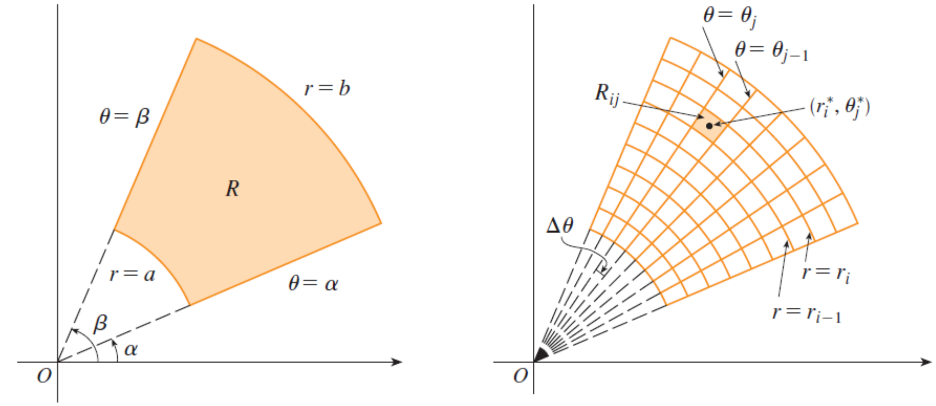
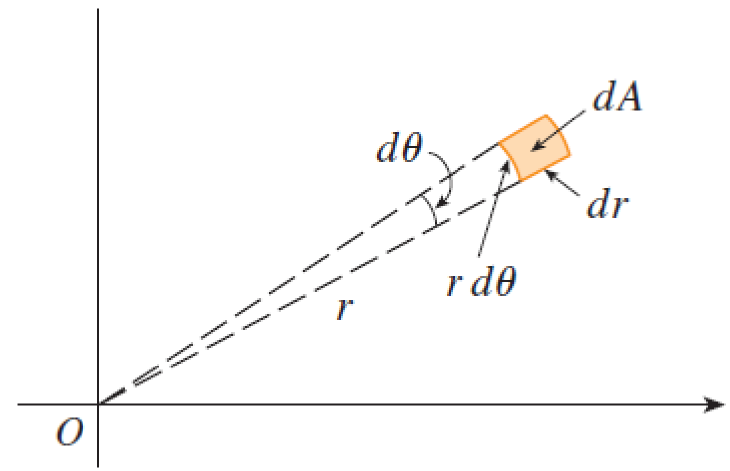


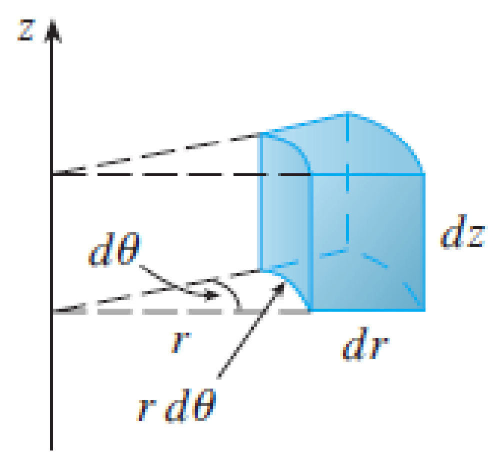
FIGURE 4



Polar







Examples:

1. Compute

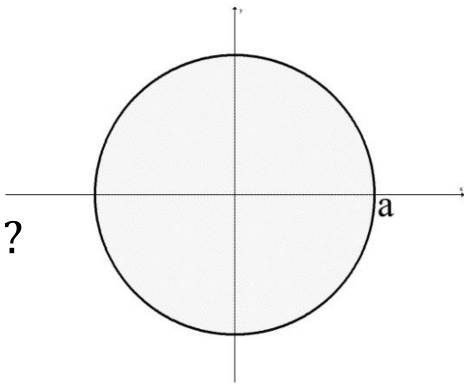
$$\iint_R \frac{\cos(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}\, dA$$

where R is the region in the first quadrant that is between $x^2 + y^2 = 49$, $x^2 + y^2 = 25$ and below $y = x$.

2. Set up the two double integrals below over the entire circular disc of radius a:

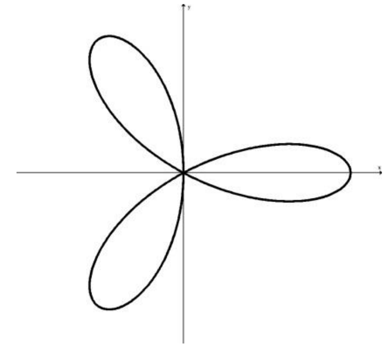
$$\iint_D 1\, dA = ?$$

$$\iint_D \sqrt{a^2 - x^2 - y^2}\, dA = ?$$



HW 15.4/5:

Find the area of one closed loop of $r = 6\cos(3\theta)$.



3. **HW 15.4/4**:

Evaluate

$$\iint_R x \, dA$$

over the region in the first quadrant between the circles $x^2 + y^2 = 16$ and $x^2 + y^2 = 4x$ using polar

Moral:

Three ways to set up a double integral:

“Top/Bottom”:

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

“Left/Right”:

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

“Inside/Outside”:

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

15.5 Center of Mass

Motivation "the see-saw"

New App: Given a thin uniformly distributed plate (a *lamina*) with density at each point $\rho(x, y)$ can we find the center of mass (*centroid*).

$\rho(x, y)$ = mass/area (kg/m^2)

We will see that

$$\overline{x} = \frac{\text{Moment about y}}{\text{Total Mass}} = \frac{\iint_R x\, p(x, y)dA}{\iint_R p(x, y)dA}$$

$$\bar{y} = \frac{\text{Moment about x}}{\text{Total Mass}} = \frac{\iint_R y\, p(x, y)dA}{\iint_R p(x, y)dA}$$

In general: If you are given ***n* points**

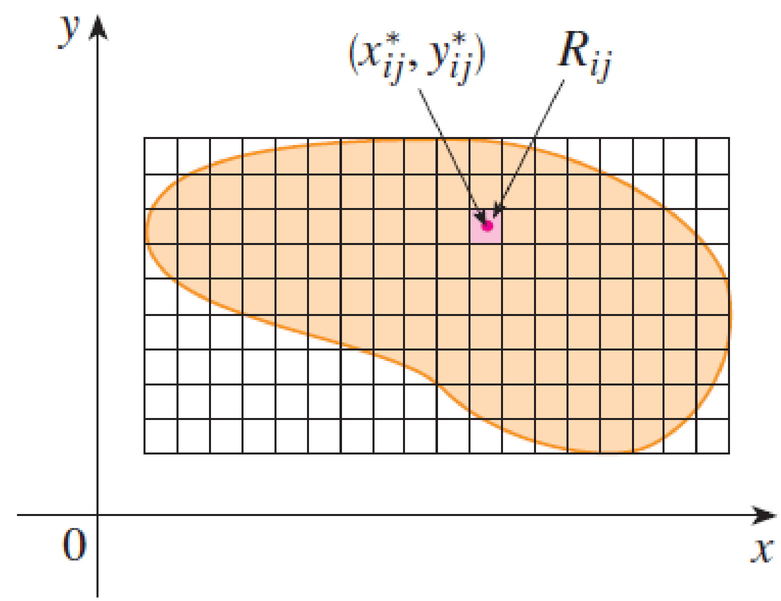
$(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ with

corresponding masses $m_1, m_2, \ldots, m_n$

then

$$\bar{x} = \frac{m_1 x_1 + \cdots + m_n x_n}{m_1 + \cdots + m_n} = \frac{M_y}{M}$$

$$\bar{y} = \frac{m_1 y_1 + \cdots + m_n y_n}{m_1 + \cdots + m_n} = \frac{M_x}{M}$$



Derivation:

1. Break region into m rows and n columns.
2. Find center of mass of each rectangle:
$$(\bar{x}_{ij}, \bar{y}_{ij})$$
3. Estimate the mass of each rectangle:
$$m_{ij} = p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A$$
4. Now use the formula for *n* points.
5. Take the limit.

$$\bar{x} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}}$$
$$= \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_{ij}\, p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}{\sum_{i=1}^{m}\sum_{j=1}^{n} p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}$$

Center of Mass:

$$\bar{x} = \frac{\text{Moment about y}}{\text{Total Mass}} = \frac{\iint_R x\, p(x,y)dA}{\iint_R p(x,y)dA}$$

$$\bar{y} = \frac{\text{Moment about x}}{\text{Total Mass}} = \frac{\iint_R y\, p(x,y)dA}{\iint_R p(x,y)dA}$$

Example:
Consider a 1 by 1 m square metal plate.
The density is given by $p(x,y) = kx$ kg/m^2 for some constant k.
Find the center of mass.

Side note: The density $p(x,y) = kx$ means that the density is proportional to x which can be thought of as the distance from the y-axis. In other words, the plate gets heavier at a constant rate from left-to-right.

Translations:

Density proportional to the dist. from…

…the y-axis -- $p(x, y) = kx$.

…the x-axis -- $p(x, y) = ky$.

…the origin -- $p(x, y) = k\sqrt{x^2 + y^2}$.

Density proportional to the square of the distance from the origin:

$$p(x, y) = k(x^2 + y^2).$$

Density inversely proportional to the distance from the origin:

$$p(x, y) = \frac{k}{\sqrt{x^2 + y^2}}$$

Example: A thin plate is in the shape of the region bounded between the circles of radius 1 and 2 in the first quadrant. The density is proportional to the distance from the origin.
Find the center of mass.